- The Gamma Function $\Gamma(x)$

For positive values of x, the Gamma function is defined in terms of the improper integral,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \; x > 0.$$

To ensure that the Gamma function is well defined, we must check to see that the improper integral converges for $x > 0$. Now,

$$\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt = I_1 + I_2$$

I_1 is improper if $x < 1$, but for $0 \le t \le 1$, $0 < e^{-t} \le 1 \Rightarrow 0 \le t^{x-1}e^{-t} \le t^{x-1}$. Therefore,
$I_1 = \lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{x-1}e^{-t}dt \le \lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{x-1}dt = \lim_{\varepsilon \to 0} \left.\frac{t^x}{x}\right|_\varepsilon^1 = \lim_{\varepsilon \to 0} \frac{1}{x}(1 - \varepsilon^x) = \frac{1}{x}$ if $x > 0$. Thus I_1 converges.

For I_2, if we let $f(t) = t^{x+1}e^{-t} \Rightarrow f'(t) = e^{-t}\left[(x+1)t^x - t^{x+1}\right]$

$= t^x e^{-t}[x + 1 - t] \equiv 0$ if $t = x + 1$.

t	x	$x+1$	$x+2$
$f\prime(t)$	$+$	0	$-$
$f(t)$	$\nearrow$	$\longrightarrow$	$\searrow$

By the first derivative test, we see that f is maximized at $t = x + 1$. So,

$I_2 = \int_1^\infty t^{x-1}e^{-t}dt = \int_1^\infty f(t)\frac{1}{t^2}dt \le f(x+1)\int_1^\infty \frac{1}{t^2}dt = f(x+1)$. This shows that I_2 also converges. We have now shown that the Gamma function $\Gamma(x)$ is well defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \; x > 0.$$

Properties of the Gamma function.

1. $\Gamma(x) > 0$ for $x > 0$.
2. $\Gamma(1) = 1$, $\Gamma(2) = 1$,$\Gamma(3) = 2$; $\Gamma(n) = (n-1)!$ for $n \in N$.
3. $\Gamma(x) = (x-1)\Gamma(x-1)$ for $x > 1$.
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proofs: 1. Integrand is positive. 2., 3. Integrate by parts. 4. Let $t = u^2$ or see later.

Example 1 $\int_0^\infty x^4 e^{-3x}dx \quad = \quad \frac{1}{3}\int_0^\infty \left(\frac{u}{3}\right)^4 e^{-u}du = \frac{1}{3^5}\Gamma(5) = \frac{4!}{3^5} = \frac{8}{81}$

$\uparrow$

$u = 3x$

Example 2 $\int_0^\infty x^2 e^{-\frac{x^2}{2}}dx \quad = \quad \int_0^\infty \sqrt{2}u^{1/2}e^{-u}du = \sqrt{2}\Gamma(3/2) = \frac{\sqrt{2}}{2}\Gamma(1/2) = \sqrt{\frac{\pi}{2}}$

$\uparrow$

$u = \frac{x^2}{2}$

Example 3 $\int_0^\infty x^4 e^{-x^3} dx \quad = \quad \frac{1}{3}\int_0^\infty u^{2/3} e^{-u} du = \frac{1}{3}\Gamma\left(\frac{5}{3}\right)$ *(Mathcad has this function built in - just like exp or sine or tangent.)*

$$\uparrow$$
$$u = x^3$$

- The Beta Function $B(m,n)$.

Although the integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is improper at $x = 0$ if $m < 1$ and improper at $x = 1$ if $n < 1$, by considering the integral in the form

$\int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^a + \int_a^b + \int_b^1 x^{m-1}(1-x)^{n-1} dx$ where $0 < a < b < 1$, the improper integrals will converge if $m, n > 0$. Therefore, we define the Beta function $B(m,n)$ by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \; m, n > 0.$$

By making the change of variable $y = 1 - x$, we can see that $B(m,n) = B(n,m)$. An alternative form for the Beta function is obtained via the substitution $x = \sin^2\theta \Rightarrow$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{(2m-1)}\theta \cos^{(2n-1)}\theta d\theta \; m, n > 0.$$

Relationship between the Beta and Gamma Functions.
The very important result that relates the Beta and Gamma Functions is:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \text{ for any } m, n > 0$$

Proof:
Consider the double integral $I = \iint_A f(x,y)\, dA$ where $f(x,y) = x^{2m-1}y^{2n-1}e^{-\left(x^2+y^2\right)}$ and A is the first quadrant $A = \{(x,y) : x > 0, y > 0\}$. Then, $I = \left(\int_0^\infty x^{2m-1}e^{-x^2}dx\right)\left(\int_0^\infty y^{2n-1}e^{-y^2}dy\right)$.
Let $x = \sqrt{t}$, then $\int_0^\infty x^{2m-1}e^{-x^2}dx = \frac{1}{2}\int_0^\infty t^{m-1}e^{-t}dt = \frac{1}{2}\Gamma(m)$. Similarly, $\int_0^\infty y^{2n-1}e^{-y^2}dy = \frac{1}{2}\Gamma(n)$. Thus,

$$I = \frac{1}{4}\Gamma(m)\Gamma(n).$$

Alternatively, if we switch to polar coordinates, we find
$I = \int_0^{\pi/2}\int_0^\infty (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} e^{-r^2} r dr d\theta$
$= \left(\int_0^{\pi/2} \sin^{(2m-1)}\theta\cos^{(2n-1)}\theta d\theta\right)\left(\int_0^\infty r^{2m+2n-1}e^{-r^2}dr\right) = \left(\frac{1}{2}B(m,n)\right)\left(\frac{1}{2}\Gamma(m+n)\right)$, i.e.

$$I = \frac{1}{4}B(m,n)\Gamma(m+n).$$

Equating these results gives the result:

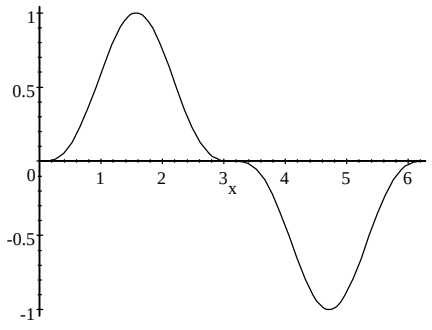
$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example 4 *If we put $m = n = 1/2$ in trig. form of Beta function, we get*
$B(1/2,1/2) = 2\int_0^{\pi/2} \sin^{(1-1)}\theta \cos^0\theta d\theta = 2\int_0^{\pi/2} d\theta = \pi \Rightarrow \{\Gamma(1/2)\}^2 = \Gamma(1)\, B(1/2,1/2) = \pi$
$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$

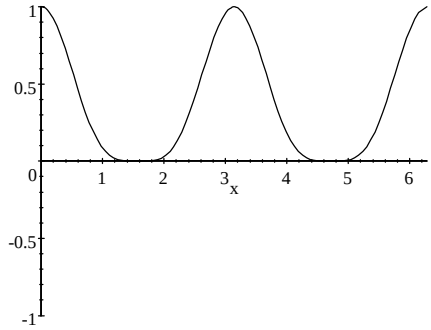
The trigonometric form of the Beta function, and the relationship with the Gamma functions allows us to evaluate integrals of the form $\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta$ without integrating. For example,

Example 5 $\int_0^{\pi/2} \sin^3\theta \cos^4\theta d\theta = \frac{1}{2}B\left(2,\frac{5}{2}\right) = \frac{1}{2}\frac{\Gamma(2)\Gamma(5/2)}{\Gamma(9/2)} = \frac{1}{2}\frac{1\cdot\Gamma(5/2)}{(7/2)(5/2)\Gamma(5/2)} = \frac{2}{35}$

Using the symmetry in the functions sin and cos over the four quadrants, we can actually use the above result for integrals over an interval which is a multiple of $[0,\pi/2]$. As an example, look at how $\sin^3 x$ and $\cos^4 x$ look like over $[0,2\pi]$.



$\sin^3 x$



$\cos^4 x$

Because of trigonometric properties ($\sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x,$ etc.) the function $\sin^3 x \cos^4 x$ possesses symmetries (as can be seen graphically) so that $\int_0^{\pi/2} \sin^3 x\cos^4 x dx = \int_{\pi/2}^{\pi} \sin^3 x\cos^4 x dx = -\int_{\pi}^{3\pi/2} \sin^3 x\cos^4 x dx = -\int_{3\pi/2}^{2\pi} \sin^3 x\cos^4 x dx$. The sign of the integrand in each quadrant is simply the product of the signs of $\sin^3 x$ and $\cos^4 x$ over the same quadrants. Once these signs have been determined, we can express a definite integral over several quadrants as a multiple of the integral over the first quadrant. For example,

Example 6 $\int_0^{\pi} \sin^3 x\cos^4 x dx = (1+1)\int_0^{\pi/2} \sin^3 x\cos^4 x dx = 2\left(\frac{2}{35}\right) = \frac{4}{35}$

Example 7 $\int_0^{3\pi/2} \sin^3 x\cos^4 x dx = (1+1-1)\int_0^{\pi/2} \sin^3 x\cos^4 x dx = 1\left(\frac{2}{35}\right) = \frac{2}{35}$

Example 8 $\int_0^{2\pi} \sin^3 x\cos^4 x dx = (1+1-1-1)\int_0^{\pi/2} \sin^3 x\cos^4 x dx = 0$

The procedure outlined here applies to all integrals of the form

$$\int_{i\pi/2}^{j\pi/2} \sin^m x\cos^n x dx$$

where i, j, m, n are integers and $m, n \geq 0$. Since all we need is the sign of the integrand, we do not need to sketch $\sin^m x\cos^n x$; a simple table of signs will be sufficient - as seen in this example.

Example 9 $\int_0^{3\pi/2} \sin^2 x \cos^5 x dx = K \int_0^{\pi/2} \sin^2 x \cos^5 x dx$. *To obtain* K, *we see that*

x	$(0, \pi/2)$	$(\pi/2, \pi)$	$(\pi, 3\pi/2)$
$\sin^2 x$	$+$	$+$	$+$
$\cos^5 x$	$+$	$-$	$-$
$\sin^2 x \cos^5 x$	$+$	$-$	$-$

$\Rightarrow K = (1 - 1 - 1) = -1$. *Therefore,*

$\int_0^{3\pi/2} \sin^2 x \cos^5 x dx = -\int_0^{\pi/2} \sin^2 x \cos^5 x dx = -\frac{1}{2} B\left(\frac{3}{2}, 3\right) = -\frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)} = -\frac{8}{105}$.

It must be stated that the procedure outlined does not apply to integrals like $\int_0^{\pi/3} \sin^2 x \cos^5 x dx$ - a non integer multiple of $\pi/2$. In this case, the usual integration techniques must be used.

EXERCISES

1. Evaluate the following:
   (a) $\Gamma(6)$: $\boxed{120}$
   (b) $\Gamma(3/2)$: $\boxed{\frac{1}{2}\sqrt{\pi}}$
   (c) $\frac{\Gamma(9/2)}{\Gamma(5/2)}$: $\boxed{\frac{35}{4}}$
   (d) $B(3, 2)$: $\boxed{\frac{1}{12}}$
   (e) $B(4, 1/2)$: $\boxed{\frac{32}{35}}$
2. Use symmetry and the Beta function to evaluate:
   (a) $\int_0^{\pi/2} \sin^3 x \cos^2 x dx$: $\boxed{\frac{2}{15}}$
   (b) $\int_0^{\pi} \sin^3 x \cos^2 x dx$: $\boxed{\frac{4}{15}}$
   (c) $\int_0^{\pi} \sin^5 x dx$: $\boxed{\frac{16}{15}}$
   (d) $\int_0^{\pi} \sin^7 x \cos^3 x dx$: $\boxed{0}$
   (e) $\int_0^{\pi} \cos^5 x dx$: $\boxed{0}$
   (f) $\int_0^{2\pi} \sin^5 x \cos^2 x dx$: $\boxed{0}$
   (g) $\int_0^{2\pi} \sin^4 x \cos^2 x dx$: $\boxed{\frac{1}{8}\pi}$